

Have we caught your interest?



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Features



Have we caught your interest?

by John H. Webb



Introduction

"Neither a borrower nor a lender be."

Today, nobody heeds the advice of Polonius to his student son Laertes. Everybody borrows and lends all the time.

- Children put their pocket money into a bank account to save up to buy a bicycle.
- Students take out loans to finance their studies.
- Credit cards are widely used for short-term borrowing.
- Young couples buy houses on mortgages of 20 to 30 years, and save for their retirement.

Understanding lending and borrowing means understanding compound interest. For a career in the world of finance, that understanding has to be broad and thorough.

In this article we will not attempt a comprehensive coverage of all the intricate details of the subject, but will try to show that compound interest

- is an essential component of numeracy,
- involves some interesting maths,
- throws up some surprising, even paradoxical results from time to time.

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A slice of history



Figure 1: The real purpose of the ziggurat?

The application of mathematics to trade and financial affairs is as old as mathematics itself. History shows that as soon as a civilisation develops writing, including the writing of numbers, the new technology is used in financial transactions and the appropriate mathematical ideas develop rapidly.

Over 4000 years ago the Sumerian merchants of Babylon and Nineveh recorded bills, receipts and promissory notes in cuneiform script on clay tablets. They drew up multiplication tables and tables of reciprocals, squares, cubes and exponentials, and used them to calculate compound interest and mortgage repayments. A clay tablet now in the Louvre tackles the problem of finding how long it will take a sum of money to double at an interest rate of 20%.

Hindu mathematicians of 500–1000 A.D. knew how to sum arithmetic and geometric series, and could solve problems involving discount and interest calculations.

There is a record of a loan agreement in London in 1183. The interest levied was 2d. in the pound per week, amounting to 43% per annum. An even higher rate was charged in 1235 by London merchants who lent money to a priory, demanding "every two months one mark for every ten marks as a recompense for losses" – an annual interest rate of 60%.

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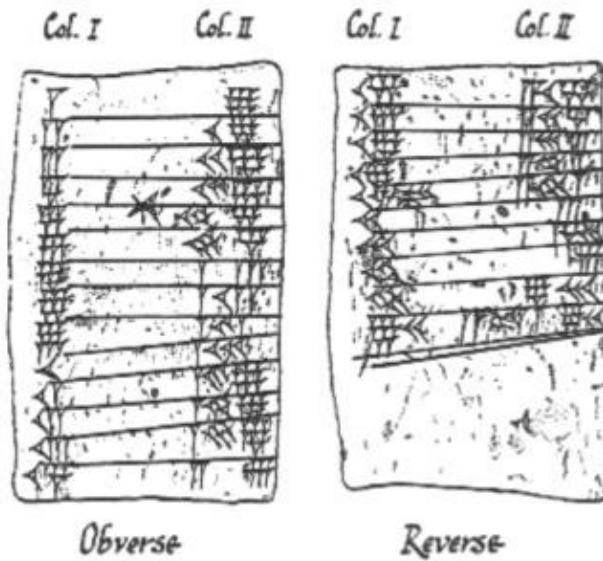


Figure 2: The 9 times table in Babylonian cuneiform

In 15th century Italy there was a tremendous increase in commercial activity. Double-entry book keeping was introduced, and textbooks in arithmetic were published: in Latin for the classical scholars in church schools, and in the vernacular for preparing students for commercial careers.

Logarithms (invented by John Napier in 1614) revolutionised financial mathematics. Only in the 20th century were they replaced by mechanical, and later electronic calculators.

Time is money



Figure 3: Time is money?

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When you borrow money from a lender such as a bank, you pay the lender *interest* – a fee for the use of the money. Interest is charged as a percentage of the *principle* (the amount of the loan) for an agreed period, usually a year.

If the principle is P and the interest rate is r , then the interest is rP . At the end of the year the loan plus the interest must be repaid: a total of $P + rP = P(1 + r)$.

If you make no repayment, your debt is now $P(1 + r)$, which must be repaid at the end of the second year with interest of $rP(1 + r)$: a total of $P(1 + r) + rP(1 + r) = P(1 + r)^2$.

After three years you will owe $P(1 + r)^3$, and after n years you will owe

$$P(1 + r)^n$$

This is the basic formula for calculating compound interest.

The same formula applies when you deposit money into a savings account. Now you are lending money to the bank, which pays you interest on your deposit. If you deposit an amount P at an interest rate r , after n years your investment will be worth $P(1 + r)^n$.

The value of money is thus seen to be time dependent. If PV denotes the *present value* of a sum of money, and FV denotes its *future value* in n years at an interest rate of r , then

$$\begin{aligned} FV &= PV(1 + r)^n \\ \text{and } PV &= FV(1 + r)^{-n} \end{aligned}$$

These simple formulae are the key to understanding complex investment and debt repayment schemes.

Frequent compounding

The basic compound interest formula assumes that interest is paid once a year. Sometimes interest is payable more frequently. A fixed deposit investment may pay interest quarterly, and car loans and mortgages are paid monthly.

If £1000 is invested at 8% for a year, it will be worth, at the end of the year

$$\begin{aligned} 1000 \times (1 + 0.08) &= 1080.00 \quad (\text{annual compounding}) \\ 1000 \times \left(1 + \frac{0.08}{4}\right)^4 &= 1082.43 \quad (\text{quarterly compounding}) \\ 1000 \times \left(1 + \frac{0.08}{12}\right)^{12} &= 1083.00 \quad (\text{monthly compounding}) \\ 1000 \times \left(1 + \frac{0.08}{52}\right)^{52} &= 1083.22 \quad (\text{weekly compounding}) \end{aligned}$$

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$$1000 \times \left(1 + \frac{0.08}{365}\right)^{365} = 1083.27 \quad (\text{daily compounding})$$

The *effective rate* of compounding 8% quarterly is 8.243%
 monthly 8.300%
 weekly 8.322%
 daily 8.327%

As the frequency of compounding increases, so does the effective rate, but successively more slowly.

In general, if interest is compounded k times a year at an interest rate of r , after t years an investment P grows to

$$P \left(1 + \frac{r}{k}\right)^{kt}$$

What happens as $k \rightarrow \infty$?

Continuous compounding

The core mathematical issue here is the behaviour of the sequence

$$\left(1 + \frac{1}{k}\right)^k$$

as $k \rightarrow \infty$.

Every calculus textbook shows that this sequence increases, but not without bound (it is not too difficult to prove that $\left(1 + \frac{1}{k}\right)^k \leq 3$ for all k). In fact, the sequence converges to a limit whose value (to 7 decimal places) is 2.7182818. This number is denoted by e . It plays a very important role in calculus.

It can then be proved that, as $k \rightarrow \infty$,

$$P \left(1 + \frac{r}{k}\right)^{kt} \rightarrow P e^{rt}$$

which is the value of an investment of P after t years at an interest rate of r , compounded continuously.

An alternative method of deriving the formula for continuously compounded interest is to set up and solve a *differential equation*.

Let $P(t)$ be the value of an investment at time t , which is earning interest at a rate r . The initial investment (when $t = 0$) is $P(0)$. Then over a short time interval from t to $t + \Delta t$, the interest earned is $P(t) \times r \Delta t$, and hence

$$P(t + \Delta t) = P(t) + P(t) \cdot r \Delta t$$

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$$\text{So } \frac{P(t + \Delta t) - P(t)}{\Delta t} = rP(t)$$

Letting $\Delta t \rightarrow 0$ gives the differential equation

$$\frac{dP}{dt} = rP$$

This is a separable differential equation, and is solved as follows:

$$\begin{aligned} & \frac{dP}{P} = rdt \\ \Rightarrow & \int \frac{dP}{P} = \int rdt \\ \Rightarrow & \ln P = rt + C \\ \Rightarrow & P(t) = e^{rt+C} \\ \text{Putting } t = 0 \text{ gives } & P(0) = e^C, \text{ and we have the solution} \\ & P(t) = P(0) e^{rt} \end{aligned}$$

which is the formula for continuously compounding interest we obtained before.

Finance and physics

The differential equation $dP/dt = rP$ describes exponential growth or decay. Similar equations arise in physics, notably in radioactive decay. This is not a chance occurrence, as there are a number of problems in financial mathematics where the underlying mathematics also models a physical process. Brownian motion describes molecular motion in a fluid, and also the random movements of the stock market. The famous Black–Scholes equation used in options and derivatives pricing has close analogies in equations describing diffusion processes in physics.

How many days in a year?

In calculating interest over a part of a year, say 137 days, it would seem natural to use the fraction $\frac{137}{365}$ (or $\frac{137}{366}$ in a Leap Year). Some financial institutions, however, use an "interest year" of 360 days. This can make a significant difference. Interest on a million pounds at 8% per annum over $\frac{137}{360}$ ths of a year is $\hat{\text{£}}30444.44$; over $\frac{137}{365}$ ths of a year is $\hat{\text{£}}30027.40$: $\hat{\text{£}}417.04$ less.

A 360–day interest year is used in several European countries and the United States. Britain and Japan use a 365–day interest year, even in a Leap Year. Some Euro–Sterling bond issues accept that there are 366 days in a Leap Year.

For a daily compounding of interest, however, it makes very little difference whether the year is taken as 360, 365 or 366 days. For example, at 8%

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$$\left(1 + \frac{0.08}{360}\right)^{360} = 1.083277435$$

$$\left(1 + \frac{0.08}{365}\right)^{365} = 1.083277571$$

$$\left(1 + \frac{0.08}{366}\right)^{366} = 1.083277596$$

On a million pounds over one year, the difference is at most 16 pence.

Double your money

How long will it take for your money to double when it is invested in a savings account at 5%? And how long will it take to double at 7%? A simple rule of thumb says that at $x\%$, money will double in $70/x$ years. So at 5%, a deposit doubles in 14 years, and at 7% it doubles in 10 years. This is the "Rule of 70".

How is the rule derived? If an amount A is invested at an interest rate of r , the problem is to determine n such that

$$2A = A(1 + r)^n$$

Some easy algebra gives

$$n = \frac{\log 2}{\log(1 + r)}$$

The log function used could be to any base, but it turns out to be a good idea to use logs to base e (natural logarithms, denoted by \ln). Then

$$n = \frac{\ln 2}{\ln(1 + r)}$$

The reason for choosing natural logarithms is that there is a representation of the function $\ln(1 + r)$ as a *Maclaurin series*

$$\ln(1 + r) = r - \frac{r^2}{2} + \frac{r^3}{3} - \dots \quad (\text{for } |r| < 1)$$

Since r is small, the terms of the Maclaurin expansion may be neglected, so that $\ln(1 + r) \approx r$. So

$$n = \frac{\ln 2}{r} \approx \frac{0.6931}{r}$$

Replacing 0.6931 by the nicer rounder number 0.7000 gives

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$$n = \frac{0.7}{r} = \frac{70}{x}$$

when $r = x\%$. The table below compares the doubling time given by the rule of 70 with the correct doubling time.

$r = x\%$	Rule of 70: $70/x$	Correct answer: $\ln 2 / \ln(1+r)$
1%	70	69.66
2%	35	35.00
5%	14	14.21
7%	10	10.24
10%	7	7.27

The Rule of 70 would be more accurate as a "Rule of 69.3". But 70 is a more manageable number. In fact, 72 has more divisors than 70, and a "Rule of 72" serves almost as well as the Rule of 70.

The rule of 70 was derived under the assumption of annual compounding. What happens with continuous compounding? Now the equation to solve is

$$\begin{aligned} 2A &= Ae^{rt} \\ \text{which gives } t &= \frac{\ln 2}{r} = \frac{0.693}{r}, \end{aligned}$$

without having to use the Maclaurin expansion.

Regular investment pays

At the beginning of each year an amount P is invested in a savings account which pays interest at a rate r . How much will be in the account at the end of n years?

The future value of the first payment is $P(1+r)^n$

The future value of the second payment is $P(1+r)^{n-1}$

The future value of the n th payment is $P(1+r)$

The total investment will therefore be worth

$$\begin{aligned} &P(1+r)^n + P(1+r)^{n-1} + \dots + P(1+r) \\ = &P(1+r) \left[1 + (1+r) + (1+r)^2 + \dots + (1+r)^{n-1} \right] \\ = &P(1+r) \frac{(1+r)^n - 1}{(1+r) - 1} \quad \text{summing the geometric series} \\ = &\frac{P(1+r)((1+r)^n - 1)}{r} \end{aligned}$$

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For example, if £1000 is invested at the beginning of each year into an account earning 10% interest, at the end of 7 years the investment will be worth

$$\frac{1000 \times 1.1 \left((1.1)^7 - 1 \right)}{0.1} = \pounds 10435.89$$

Early investment pays

When Alice and Bob were both 19 years old they started new jobs. Alice started a savings programme, investing £1000 at the beginning of each year at 10% per annum. At the end of 7 years her account shows a total of £10435.89 (see the calculation above).

Bob, who has not saved a penny until now, is impressed by the size of Alice's nest egg, and decides to start investing in the same way. Alice decides not to put any more into her account, just letting it grow at 10% per annum. Bob is determined to build up his investment until it is bigger than Alice's. How long will it take for Bob's investment to overtake Alice's?

The astonishing answer is that it will take 32 years!

If you find this hard to believe, check the calculations below.

After 31 years, Alice's investment will be worth

$$10435.89 \times (1.1)^{31} = 200310.05,$$

Bob's investment will be worth

$$\frac{1000 \times 1.1 \times \left((1.1)^{31} - 1 \right)}{0.1} = 200137.77,$$

and Alice is still ahead of Bob.

After 32 years,

Alice's investment will be worth

$$10435.89 \times (1.1)^{32} = 220341.05,$$

Bob's investment will be worth

$$\frac{1000 \times 1.1 \times \left((1.1)^{32} - 1 \right)}{0.1} = 221251.54$$

and Bob has at last overtaken Alice.

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Repayment of loans

When you buy a house or a car, you borrow the purchase price from a bank, and agree to pay the money back in regular equal instalments over a certain time. If the amount of the loan is A , the interest rate is r and the number of repayments is n , what is the amount of the repayment R ?

The key is to look at the problem from the point of view of the bank: their loan A must be balanced by the present value of the repayments.

The present value of the first repayment R is $R(1+r)^{-1}$

the present value of the second repayment R is $R(1+r)^{-2}$

and, in general

the present value of the n th repayment R is $R(1+r)^{-n}$

The total present value of all the repayments is therefore

$$R(1+r)^{-1} + R(1+r)^{-2} + \dots + R(1+r)^{-n}$$

which is a geometric series of n terms with first term $R(1+r)^{-1}$ and common ratio $(1+r)^{-1}$. The sum of the series is

$$\begin{aligned} & \frac{R(1+r)^{-1} \left(1 - (1+r)^{-n}\right)}{1 - (1+r)^{-1}} \\ = & \frac{R \left((1+r)^n - 1\right)}{r(1+r)^n} \end{aligned}$$

When this is equated to A , we obtain

$$R = \frac{Ar(1+r)^n}{(1+r)^n - 1}$$

For example, a loan of $\hat{\text{£}}100\,000$ is to be repaid over 25 years in equal monthly instalments. If the annual interest rate is 7%, what is the monthly repayment?

Here $A = 100000$, $r = 0.07/12$ and $n = 300$ (since the repayments are monthly). So

$$\begin{aligned} R &= \frac{100000 \times \frac{0.07}{12} \times \left(1 + \frac{0.07}{12}\right)^{300}}{\left(1 + \frac{0.07}{12}\right)^{300} - 1} \\ &= 706.78 \end{aligned}$$

The repayments are made up of two components: interest on the outstanding balance, and an amount of capital repayment. The table below shows the breakdown of the first three repayments.

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	Interest	Capital	Outstanding
		Repayment	Capital
1	583.33	123.45	99876.55
2	582.61	124.17	99752.38
3	581.89	124.89	99627.49

Initially, the monthly repayment is largely interest, with only 17.46% of capital repayment. The interest component falls and the capital repayment rises as the payments are made.

After 20 years, what proportion of the capital has been repaid? The answer could be found by continuing the above table for 240 lines, an easy exercise on a spreadsheet. A more elegant solution is to realise that after 240 payments, the outstanding capital is precisely the present value of the remaining 60 payments:

$$\begin{aligned}
 & R(1+r)^{-1} + R(1+r)^{-2} + \dots + R(1+r)^{-60} \\
 = & \frac{R}{r} \left(1 - (1+r)^{-60} \right) \quad (\text{summing the geometric series}) \\
 = & \frac{706.78}{\frac{0.07}{12}} \left(1 - \left(1 + \frac{0.07}{12} \right)^{-60} \right) \\
 = & 35693.80
 \end{aligned}$$

So more than a third of the capital is still outstanding, even after 80% of the repayments have been made.

Suppose that, for purely numerical neatness, it is decided to round the monthly repayment up from £706.78 to £710. How does this affect the number of repayments?

From the formula relating A , r , n and R

$$A = \frac{R((1+r)^n - 1)}{r(1+r)^n}$$

we obtain

$$n = \frac{\ln R - \ln(R - Ar)}{\ln(1+r)}$$

and thence, putting $R = 710$, $A = 100000$ and $r = 0.07/12$ we obtain

$$n = 296.35$$

So a substantial saving of more than 3 repayments is made with a very small "rounding up". Instead of 300 payments of £706.78, totalling £212034, 296.35 payments of £710 are made, totalling £210408.50. The difference is £1625.50!

Rounding the repayments up to £720 per month saves a further 10 repayments at the end. Check the figures yourself!

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Compound interest deception

As we have seen more than once, compound interest can produce surprises. The unexpected way in which compound interest works can be turned around by dishonest advertisers. Here is an example.

A loan of £2000 is offered at "only 5%", to be repaid over a year in monthly instalments. The catch lies in the incorrect, but plausible way in which the instalments are calculated.

"At 5% per annum, the interest on £2000 is £100, making a total of £2100. So the monthly repayments are $£2100 \div 12 = £175$."

A closer look reveals that the true interest rate is a little over 9%. To find the true interest rate, we use the formula

$$A = \frac{R((1+r)^n - 1)}{r(1+r)^n}$$

with $A = 2000$, $R = 175$, and $n = 12$. The monthly rate, r , is the unknown. Putting $1+r = x$ and multiplying up gives

$$2000(x-1)x^{12} = 175(x^{12} - 1)$$

which simplifies to

$$80x^{13} - 87x^{12} + 7 = 0$$

(An obvious solution is $x = 1$, which corresponds to $r = 0$, clearly not a solution of the original problem. But then we recall that we multiplied through by x , and that is how this solution came about.) So we must find a solution of

$$80x^{13} - 87x^{12} + 7 = 0$$

other than $x = 1$. Some trial and error with a calculator shows that $x = 1.0075872$ is a solution, corresponding to an interest rate of 0.75872% per month, or a little over 9.1% per annum.

Simon Stevin's Problem



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Figure 4: Simon Stevin's statue in Bruges

The mathematician Simon Stevin of Bruges (1548–1620) is famous for introducing decimal fractions into arithmetic. In a textbook of commercial arithmetic published in 1585 he set and solved the following problem:

A debt of $\text{£}1500$ per annum, to be paid off over the next 22 years, is settled by a single payment of $\text{£}15\,300$. What was the rate of interest?

His method of solution was to refer to interest tables to find the interest rate which best fitted the figures, which turned out to be 8%. This was the only effective method of solving such problems until the advent of calculators and computers.

In mathematical terms, Stevin's problem is the same as the problem in the previous section, so we will end this introduction to compound interest with the words of many a maths textbook: "the details are best left to the reader".

About the author



John H. Webb was born in Cape Town and studied mathematics at the University of Cape Town. He won a scholarship to Cambridge where he obtained a Ph D. Back at the University of Cape Town, his career as a research mathematician was eventually overtaken by interests in mathematics education, with particular emphasis on popularising mathematics and identifying and stimulating promising students.

He edits Mathematical Digest, a quarterly magazine for high schools, runs a maths competition for schools in the Cape Town area, and directs a nationwide Mathematical Talent Search, a problem-solving programme by correspondence which selects and trains South African teams for the International Mathematical Olympiad.

He is at present on sabbatical leave, and is spending six months working with the Millennium Mathematics Project in Cambridge. His visit is supported by the Institute of Actuaries.



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